Automatic Control of Fluid Flow

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This paper illustrates the use of analytical methods for the design of a flow-control system. Linearized equations are derived and a complete analysis is made of the control of the system. The effect of controller modes and process time constants is investigated. The calculations show that there is an optimum value of the process time constant for optimum response.

DERIVATION OF EQUATIONS

In the chemical industries the automatic control of fluid flow is encountered perhaps more often than the control of any other single variable. The successful control of fluid flow is not simple, primarily because of the small time constants involved in the flow process. This paper will analyze flow control in detail and will point out some of the factors where difficulty may be encountered.

The System

The most commonly used method of controlling flow is to insert a flow meter and control valve into the flow line. The signal from the flow meter actuates a controller, which in turn operates the control valve.

The analysis of the system requires the derivation of the dynamic equations for each part of the system. The equations which truly represent the flow process are nonlinear. Owing to the difficulty of solving nonlinear equations without a computer, only linearized equations will be used in this paper. This procedure is justified because the analysis is simplified and the methods of analysis for linear equations can be applied. Furthermore much valuable information can be obtained from linear equations, and in many instances the responses calculated will check the experimental results if the disturbance is not too large.

An important feature of the analytical procedure to be presented is that an over-all view of the system operation is obtained, allowing the determination of the effect of each of the system parameters upon both the transient and frequency responses.

The Process

The process under consideration is the pipeline and control valve. For any given system two variables, the pressure drop and the valve opening, determine the rate of flow of the fluid. The exact equation describing the flow is rather complicated. Many system properties such as pipe length and volume, compressor or pump characteristics, and flow characteristics and fluid inertia have a part in fixing the time constants of the process. The time constants of the process and the other parts of the system determine the types of responses the system will have. The limiting case occurs when the response to change in pressure drop or valve opening is instantaneous. The process time constant is then 0. In the first part of this paper the system will be considered to have a zero time constant, and then the effect of a finite process time constant will be investigated.

A linear process with a 0 time constant will have a transfer function

$$G_p = K_p \tag{1}$$

The same process with a simple resistance and capacitance will have a transfer function

$$G_p = \frac{K_p}{T_p s + 1} \tag{2}$$

The Controller

Usually in the design of a control system all parts of the system are fixed except for the controller modes and the controller parameters. The equations for the ideal controller modes are easily written. There is another important characteristic of controllers which must be considered and that is the time lag in the controller mechanism. In most control systems time lags other than the process time lag may be neglected. However in flow control, with small process time lags, the lags in the other parts of the system become significant. Unfortunately there is little published information on the dynamic characteristics of industrial controllers. Information of this type would be very helpful to control engineers. A time constant of 1 sec. will be assumed. The controller transfer functions then are proportional control

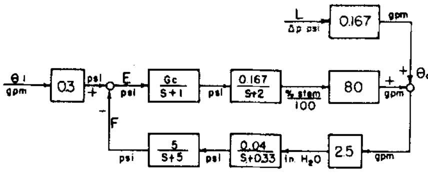

Fig. 1. Block diagram showing specific transfer functions.

$$G_c' = \frac{K_c}{s + 1} \tag{3}$$

proportional plus integral control

$$G_c' = \frac{K_c[1 + 1/(T_i s)]}{s + 1}$$

$$= \frac{K_c(s + 1/T_i)}{s(s + 1)} = \frac{K_c(s + z)}{s(s + 1)} \tag{4}$$

and proportional plus integral plus rate control (three mode)

$$G_c' = \frac{K_c[1 + 1/(T_i s) + T_r s]}{s + 1}$$

$$= \frac{K_c(T_r T_i s^2 + T_i s + 1)}{T_i s(s + 1)}$$

$$= \frac{K_c T_r (s + z_1)(s + z_2)}{s(s + 1)} \tag{5}$$

The Transmitter

The transmitter is the mechanism which converts the differential pressure of the flowmeter to a pneumatic pressure in the range of 3 to 15 lb./sq. in. There is again little published information on the dynamic response of transmitters. The author has seen some information that suggests that a reasonable time constant is 3 sec. The transfer function of the transmitter then is

$$G_{tr} = \frac{K_{tr}}{3s + 1} \tag{6}$$

The Pneumatic Tubing

In any pneumatic control system, tubing is used to transmit signals from the measuring means to the controller and to the control valve. A study of the several papers published on the dynamic response of pneumatic tubing (*1*, *2*, *3*) indicates that the response can be accurately represented only by rather complex equations. Furthermore, under some conditions the frequency-response curves are somewhat cyclic. The action of a control system with tubing under these conditions would be very peculiar. In order to simplify the analysis, it will be assumed that the tubing is of such diameter and length that the response is similar to that of a first-order system over the range of frequencies to be used and that the time constant is 0.2 sec.

The transfer function then is

$$G_{tu} = \frac{K_{tu}}{0.2s + 1} = \frac{5K_{tu}}{s + 5} \quad (7)$$

The Control Valve

Most control valves have a dynamic action that can be represented by a second-order equation; however the second time constant is small and can usually be neglected. A time constant of 0.5 sec. will be assumed; then

$$G_v = \frac{K_v}{0.5s + 1} \quad (8)$$

Operating Conditions

The following numerical values have been selected for the flow system to be discussed:

Control Point: 40 gal./min.

Flowmeter: at a flow of 40 g./min. $\Delta h_m = 50$ in. of water. The flow–pressure-drop relationship is parabolic.

Process: time constant of 0. At control point the pressure drop across the valve is 120 lb./sq. in.

Control Valve: at the control point the valve is 50% open, and the valve top pressure is 9 lb./sq. in. The flow-valve stem relationship is linear. The time constant is 0.5 sec.

Transmitter: at the control point the output is 9 lb./sq. in. The time constant is 3 sec.

Tubing: the time constant is 0.2 sec.

Controller: in addition to the normal controller functions, the controller has a time constant of 1 sec.

System Transfer Functions

Controller

If G_c is the controller mode, then

$$G_c' = \frac{G_c}{s + 1} \, \frac{\text{lb./sq. in.}}{\text{lb./sq. in.}} \quad (9)$$

Valve

Since the valve is 50% open when the valve top pressure is 9 lb./sq. in. and the valve is a linear valve,

$$K_v = 0.5/6$$

$$= 0.0833 \, \frac{\%\text{ stem position}/100}{\text{lb./sq. in.}}$$

and

$$G_v = \frac{0.0833}{0.5s + 1} = \frac{0.167}{s + 2} \cdot \frac{\%\text{ stem position}/100}{\text{lb./sq. in.}} \quad (10)$$

Transmitter

The output of the transmitter is 9 lb./sq. in. when the input is 50 in. of water. Then

$$K_{tr} = 6/50 = 0.12 \, \frac{\text{lb./sq. in.}}{\text{in. water}}$$

and

$$G_{tr} = \frac{0.12}{3s + 1} = \frac{0.04}{s + 0.333} \cdot \frac{\text{lb./sq. in.}}{\text{in. water}} \quad (11)$$

Tubing

The time constant is 0.2 sec., and there is no change of units; therefore

$$G_{tu} = \frac{1}{0.2s + 1} = \frac{5}{s + 5} \cdot \frac{\text{lb./sq. in.}}{\text{lb./sq. in.}} \quad (12)$$

Process, valve stem position

The flow of 40 gal./min. occurs when the valve is 50% open and the valve is linear; then

$$K_p = 40/0.5 = 80 \cdot \frac{\text{gal./min.}}{\%\text{ stem position}/100} \quad (13)$$

Process, pressure drop

The flow–pressure-drop relationship is parabolic; therefore K_p' is the slope of the flow curve at 40 gal./min., where $Q = c\sqrt{\Delta_p}$ and $c = 40/\sqrt{120} = 3.66$; then

$$G_L = K_p' = \frac{dQ}{d\Delta p} = 0.167 \, \frac{\text{gal./min.}}{\Delta p} \quad (14)$$

Another conversion factor needed is one to change units from flow in gallons per minute to differential in inches of water. The relationship is parabloic so that $c = 5.66$ when $Q = 40$ and $\Delta h_m = 50$. Then

$$\frac{dQ}{d\,\Delta h_m} = 0.4 \, \frac{\text{gal./min.}}{\text{in. water}} \quad (15)$$

Likewise the control-point index indicates gallons per minute, but the signal to the controller is pounds per square inch; therefore the conversion factor at the control point is

$$K = 0.12 \times 2.5 = 0.3 \, \frac{\text{lb./sq. in.}}{\text{gal./min.}} \quad (16)$$

The block diagram for the system with all the transfer functions inserted in the blocks is shown in Figure 1.

The System Equations

The open-loop equations for this system are

$$\frac{\theta_0}{E} = \frac{0.167 \times 80G_c}{(s + 1)(s + 2)} = \frac{13.333G_c}{(s + 1)(s + 2)} \quad (17)$$

$$\frac{F}{\theta_0} = \frac{2.5 \times 0.04 \times 5}{(s + 0.333)(s + 5)} = \frac{0.5}{(s + 0.333)(s + 5)} \quad (18)$$

The closed-loop equations are

$$\frac{\theta_0}{\theta_i} = \frac{0.3\theta_0/E}{1 + (\theta_0/E)(F/\theta_0)} = \frac{4G_c(s + 0.333)(s + 5)}{(s + 0.333)(s + 1)(s + 2)(s + 5) + 6.67G_c} \cdot \frac{\text{gal./min.}}{\text{gal./min.}} \quad (19)$$

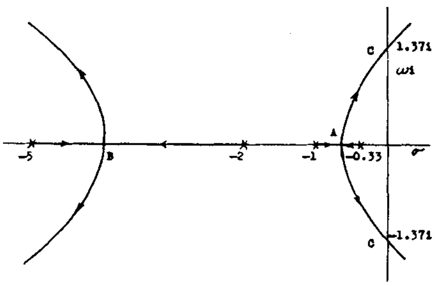

Fig. 2. Root-locus diagram with proportional control.

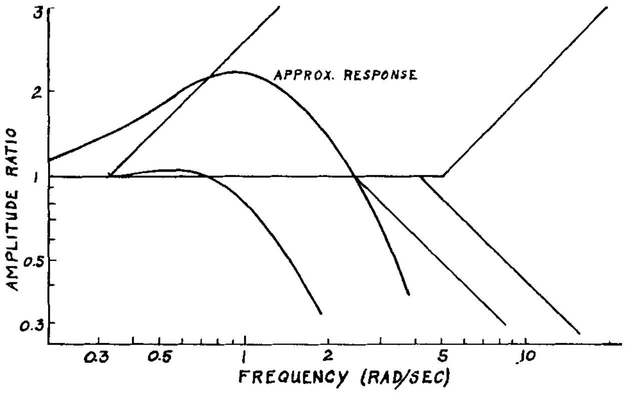

Fig. 3. Approximate frequency response with porportional control.

$$\frac{\theta_0}{L} = \frac{G_L}{1 + (\theta_0/E)(F/\theta_0)} = \frac{0.167(s + 0.333)(s + 1)(s + 2)(s + 5)}{(s + 0.333)(s + 1)(s + 2)(s + 5) + 6.67G_c} \cdot \frac{\text{gal./min}}{\Delta p} \quad (20)$$

Equations (19) and (20) are the linear equations representing the dynamic action of the flow control system. If nonlinear effects are to be investigated, the proper equations must be derived and the solutions obtained by an electronic computer.

The procedure for the selection of the modes of control and the controller parameters will be found in the following sections.

The modes of control and the specific controller parameters selected will determine the responses of the system to any input. The simplest type of controller is the proportional controller; therefore it should be studied to determine its effectiveness in controlling the given flow process. The transfer function for the proportional controller is given by Equation (3). Equations (19) and (20) then become

$$\frac{\theta_0}{\theta_i} = \frac{4K_c(s + 0.333)(s + 5)}{(s + 0.333)(s + 1)(s + 2)(s + 5) + 6.67K_c} \quad (21)$$

$$\frac{\theta_0}{L} = \frac{(s + 0.333)(s + 1)(s + 2)(s + 5)}{(s + 0.333)(s + 1)(s + 2)(s + 5) + 6.67K_c} \quad (22)$$

PROPORTIONAL CONTROL

The root-loci method (*5*, *6*) will be used to analyze the system. A method estimating the effect of each of the system parameters upon the responses (*4*) is very helpful in the analysis. It will be shown that a great deal of information can be obtained about the system with very little calculation.

In the previous section the dynamic equations for the flow system were derived. Equation (20) can be written without the constant in the numerator, making the ratio θ_0/L dimensionless. Inspection of the derivation shows that a change in flow of 1 gal./min. is caused by a change in Δp of 6 lb./sq. in. The equation will be used in this form.

The root-locus diagram is a plot of the roots (closed-loop poles) of the denominators of Equations (21) and (22) as a function of K_c. Figure 2 shows the root-locus diagram for this system with proportional control. The points marked x are the poles of the product $\theta_0/E \times F/\theta_0$. The arrows indicate the direction of increasing K_c as K_c goes from 0 to ∞. The approximate curves can be drawn without any calculation. (Rules for the construction of root-loci diagrams are given in references *5* and *6*.) The curves show that the system is overdamped; that is the closed-loop poles are real and negative as K_c goes from 0 to some value at which point A or B is reached. As K_c increases, the roots become complex and the system is then underdamped. As the closed-loop poles on the right branch pass the point C, the system becomes unstable. The exact location of the curves can be calculated, but that is not necessary at this time. The one value of interest is the value of K_c at the point C, which is found to be 4.52.

The final or steady state values of θ_0/θ_i and θ_0/L after a step change are found from the final value theorem and are

$$\left(\frac{\theta_0}{\theta_i}\right)_{ss} = \frac{4 \times 5 \times 0.333K_c}{1 \times 2 \times 5 \times 0.333 + 6.67K_c} = \frac{2K_c}{1 + 2K_c} \quad (23)$$

$$\left(\frac{\theta_0}{L}\right)_{ss} = \frac{1 \times 2 \times 5 \times 0.333}{1 \times 2 \times 5 \times 0.333 + 6.67K_c} = \frac{1}{1 + 2K_c} \quad (24)$$

When $K_c = 4.52$

$$\left(\frac{\theta_0}{\theta_i}\right)_{ss} = \frac{2 \times 4.52}{1 + 2 \times 4.52} = 0.869 \quad (25)$$

$$\left(\frac{\theta_0}{L}\right)_{ss} = \frac{1}{1 + 2 \times 4.52} = 0.096 \quad (26)$$

These values indicate that if the control point is changed, the steady state value reached will be only 86.9% of the change, or the error will be 13.1%. Likewise if a change in L is made, the steady state error will be 9.6% of the change. Actually the system would not be operated with $K_c = 4.52$, since it would be at the point of instability. A smaller value of K_c such as 2 would cause the steady state errors to be even greater.

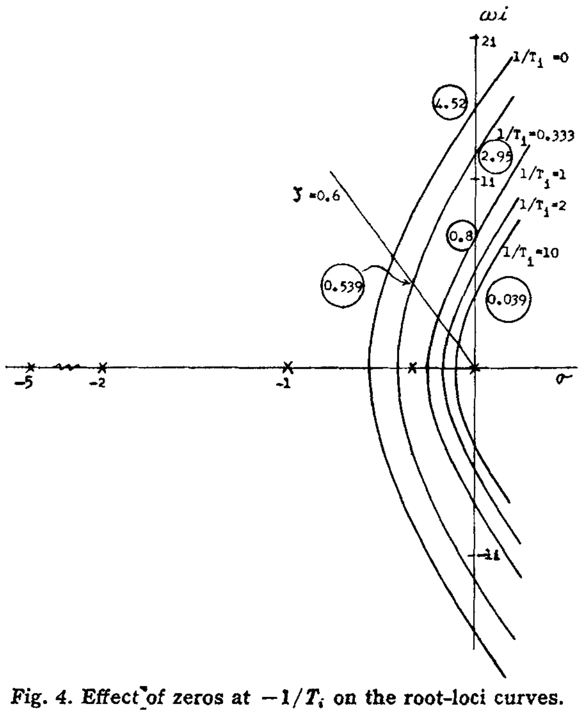

Fig. 4. Effect of zeros at $-1/T_i$ on the root-loci curves.

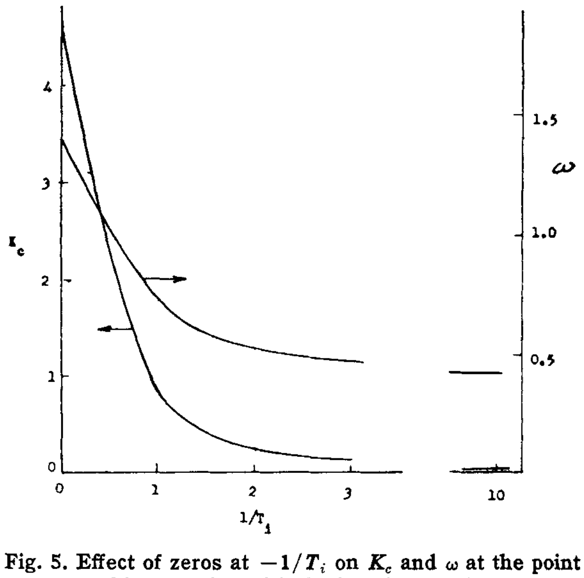

Fig. 5. Effect of zeros at $-1/T_i$ on K_c and ω at the point of intersection with the imaginary axis.

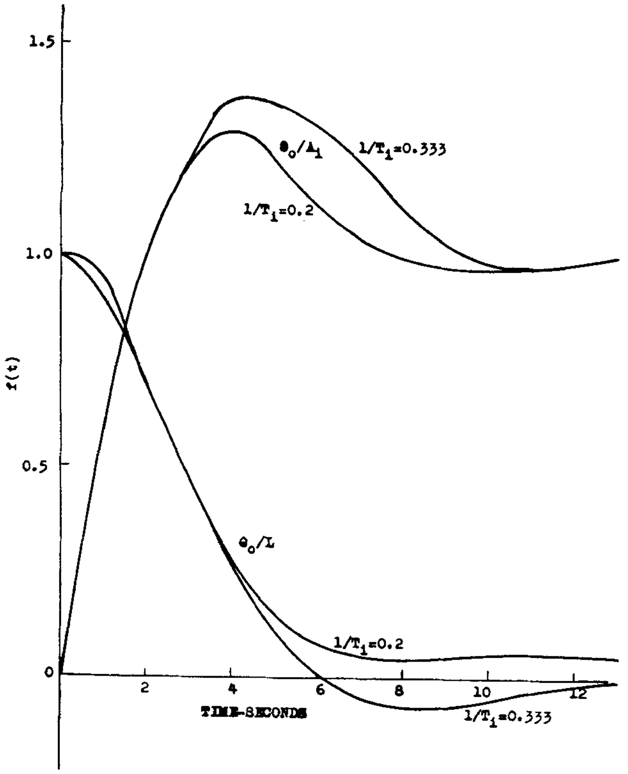

Fig. 6. Transient responses after step inputs.

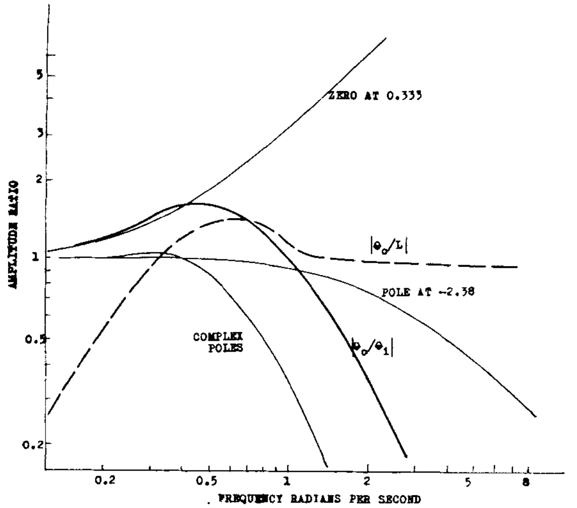

Fig. 7. Frequency responses.

Whether or not these steady state errors are too large depends upon the process requirements. If one assumes that the errors are too large, it is obvious that proportional control is not satisfactory and consequently other modes of control must be used.

The value of the root-loci diagram can be illustrated at this point. Figure 2 shows the individual time constants of the system and their effect upon the poles of the closed-loop functions. Any closed-loop poles far to the left will have little effect upon responses. These poles produce terms of the form Ae^{-pt} in the transient response, where if p is large the term becomes 0 in a very short time. It is the poles close to the origin that control the response. The poles close to the origin in this system are present because of the time constants of the controller and transmitter. In other words, the control action is limited not by the process lag but by the slow action of the transmitter and controller.

It is customary to choose the conjugate complex poles which control the response so that the damping coefficient is in the neighborhood of 0.6 to 0.7. This selection then fixes the values of the closed-loop poles. The exact values of these poles can be easily calculated when necessary. The transient response following a step input of θ_0/θ_i is determined almost entirely by the complex poles and the zero at –0.333 [Equation (21)]. The response will be that of a normal second-order system modified by the zero, which will tend to make the response much more cyclic.

The frequency response is easily approximated as shown in Figure 3. The natural frequency can be obtained by direct measurement from Figure 2. Only the asymptotic lines are shown for the first-order terms, since only an approximation is desired. For the second-order term the peak occurs at $\omega_s = 0.53$ and has a value of 1.04; the curves pass through the ordinate 0.707 when $\omega_s = 1.15$. Up to a frequency of about 0.4 radian/sec. the response is close to 1. As the frequency increases to about 0.8 radian/sec., the amplitude ratio increases to over 2, an undesirably high value.

A similar analysis may be made of Equation (22) for the ratio θ_0/L. Equations (21) and (22) have the same poles, but Equation (22) has two more zeros. The zeros and poles far to the left have little effect upon the transient response. The zeros at -1 and -0.333 will cause a very cyclic response. The frequency response can be approximated graphically as shown above. Owing to the zeros at -1 and -0.333 the frequency response will be high over a wide range. The ratio $|\theta_0/L|$ should approach 0 quickly in a well-controlled system.

From the approximate analysis made, it is quite positive that proportional control will not produce satisfactory results for this system. Therefore the use of other modes of control must be investigated.

PROPORTIONAL PLUS INTEGRAL CONTROL

Since proportional control proved to be unsatisfactory, the next step is to investigate the effect of the combination proportional plus integral control. It is known that the addition of integral control will cause the steady state error to be 0, and it also may have some other beneficial effects.

The controller transfer function is given be Equation (4). The product of the forward and feedback transfer functions from Equations (17) and (18) is

$$\frac{\theta_0}{E} \times \frac{F}{\theta_0} = \frac{6.67K_c(s + 1/T_i)}{s(s + 0.333)(s + 1)(s + 2)(s + 5)} \qquad (27)$$

The closed-loop equations then are

$$\frac{\theta_0}{\theta_i} = \frac{4K_c(s + 0.333)(s + 5)(s + 1/T_i)}{s(s + 0.333)(s + 1)(s + 2)(s + 5) + 6.67K_c(s + 1/T_i)} \qquad (28)$$

$$\frac{\theta_0}{L} = \frac{s(s + 0.333)(s + 1)(s + 2)(s + 5)}{s(s + 0.333)(s + 1)(s + 2)(s + 5) + 6.67K_c(s + 1/T_i)} \qquad (29)$$

The integral mode of control has added the terms which contain the term $1/T_i$. The quantity $1/T_i$ may theoretically have any value from 0 to ∞ but is actually limited by range built into the controller. With $1/T_i = 0$ the control is proportional only, and with $1/T_i = \infty$ the integral action is so fast that the controller acts in the same manner as a two-position controller. Therefore as the zero at $-1/T_i$ moves away from the origin, the speed of the integral action is increasing.

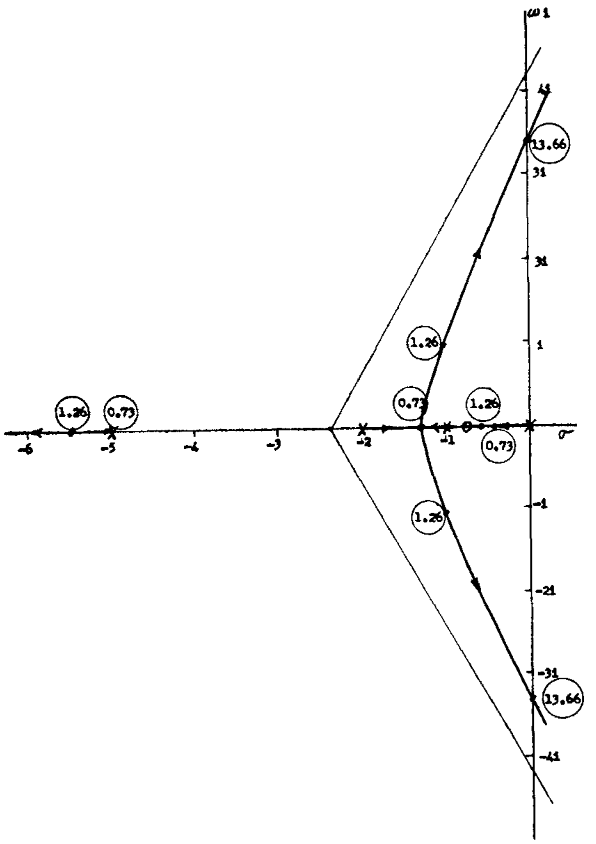

Fig. 8. Root-locus diagram with three-mode control; $z_1 = 0.333$, $z_2 = 0.8$.

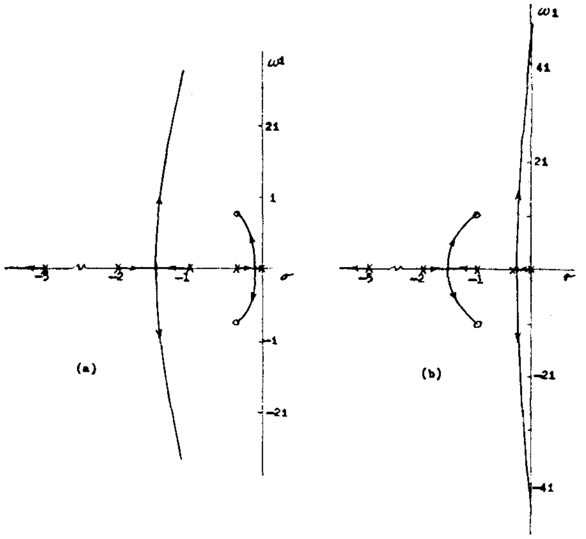

Fig. 9. Root-loci diagrams with complex zeros; (*a*) complex zeros at -0.375 ± 0.75, (*b*) complex zeros at $-1 \pm i$.

The approximate root-loci diagrams for several values of $1/T_i$ are shown in Figure 4 (the zeros at $-1/T_i$ are not shown). The numbers within the circles are the values of K_c at the point. Very few calculations are needed to sketch the curves in the form shown (*6*). The effect of changing the values of $1/T_i$ upon the closed-loop poles is at once obvious. When these approximate values are substituted into Equations (28) and (29), one can make a reasonable estimate of the responses.

Specifically as $1/T_i$ increases, it is seen that the branched part of the curve contracts and the value of K_c, when the loci curve intersects the imaginary axis, decreases. This change indicates that the system is becoming less stable and becomes unstable at smaller values of K_c. Figure 5 shows the effects more clearly where K_c and ω drop rapidly as

$$\frac{\theta_0}{\theta_i} = \frac{4K_c(s + 0.333)(s + 5)}{s(s + p_3)(s + p_4)(s + \sigma + \omega i)(s + \sigma - \omega i)} \tag{30}$$

$1/T_i$ increases. Actually the control system should not be operated at the point of intersection of the loci curve and the imaginary axis. Often a suitable operating point is that which gives a damping coefficient of about 0.6. The *line representing* $\zeta = 0.6$ is shown in Figure 4, and it is seen that K_c at this point is much less than at the point of intersection of the loci curve and the imaginary axis. From a practical standpoint there is a limit to how small K_c can become, since most industrial controllers have a maximum proportional band of 150 to 200%. This means that K_c must be greater than 0.5. Therefore the present system could not be operated with $1/T_i$ greater than about 0.333, for under these conditions with $\zeta = 0.6$, $K_c = 0.539$ (proportional band 186%).

It so happens that when $1/T_i = 0.333$, one of the poles in Equations (27) is canceled, simplified the equations. Therefore this a convenient selection of $1/T_i$ for which the transient and frequency response of the system will be determined. Then for a step change in θ_i the ratio θ_0/θ_i is

The poles p_3 and p_4 will have values between -2 and -5; the other two terms represent the conjugate complex poles, the values of which depend upon the value of the damping coefficient selected. The position of the poles p_3 and p_4 will not be greatly changed by any change made in the damping coefficient. The position of the poles can be calculated by any one of several methods (*6*).

When $1/T_i = 0.333$ and $\zeta = 0.6$, then $p_3 = 2.38$, $p_4 = 4.96$, $\sigma = 0.333$, $\omega = 0.44$, $\omega_n = 0.56$, and $K_c = 0.539$. Equation (30) then becomes for a step change in θ_i

$$\frac{\theta_0}{A_i} = \frac{4 \times 0.539(s + 0.333)(s + 5)}{s(s + 4.96)(s + 2.38)(s + 0.333 + 0.44i)(s + 0.333 - 0.44i)} \tag{31}$$

The terms $s + 5$ and $s + 4.96$ are so nearly equal that they will be canceled.

The transient response is shown in Figure 6. The response is relatively fast, but the maximum overshoot of 38% is perhaps too large. If a smaller overshoot is desired, the only change that can be made is to move the zero at $-1/T_i$ nearer to the origin. For example if $1/T_i = 0.2$, the rate of rise is not changed but the overshoot is now about 30%. From the root-loci diagram it can be seen that moving the zero at $-1/T_i$ nearer to the origin produces a closed-loop zero at $-1/T_i$ and pole between the origin and $-1/T_i$. The pole being nearest the origin will exert the greater effect upon the response; however the combination of a pole and zero close together will not have an appreciable effect.

The ratio θ_0/L with $1/T_i = 0.333$ and $\zeta = 0.6$ is investigated. Then for a step change in the pressure drop Equation (29) becomes

$$\frac{\theta_0}{A_L} = \frac{s(s + 1)(s + 2)(s + 5)}{s(s + 4.96)(s + 2.38)(s + 0.333 + 0.44i)(s + 0.333 - 0.44i)} \tag{32}$$

Again the terms $s + 5$ and $s + 4.96$ are canceled. The curves have an initial value of 1 when $t = 0+$ because the

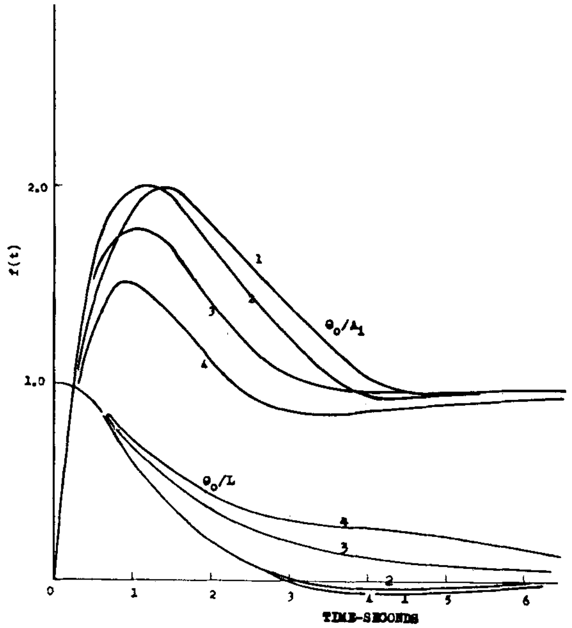

Fig. 10. Transient responses after a step input.

Curve	z_1	z_2
1	0.333	1.0
2	0.333	0.8
3	0.333	0.6
4	0.333	0.333

number of zeros and poles are equal in Equation (32). Moving $1/T_i$ nearer to the origin will not change the curve at the initially high start. This can be accomplished only by the addition of a pole in the closed-loop equation so that there will be an excess of poles over the zeros. As the move toward the origin takes place, a pole near the origin is introduced, resulting in a slow response. The curve for $1/T_i = 0.2$ is shown in Figure 6, where it is seen that the curve is less cyclic but that an error tends to persist for a longer time. This effect becomes more pronounced as $1/T_i$ approaches the origin.

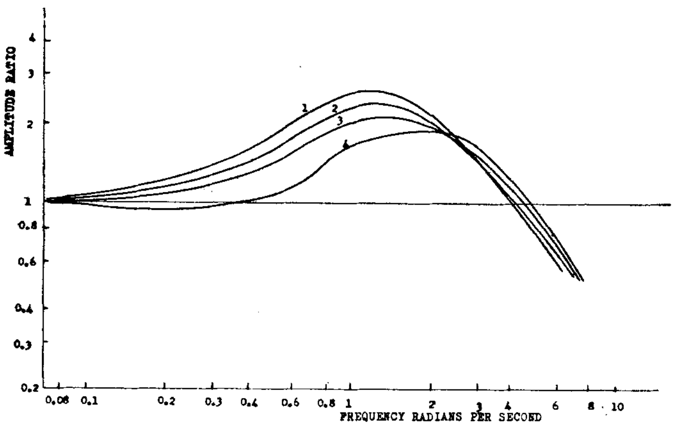

Fig. 11. Frequency response-control point change.

Curve	z_1	z_2
1	0.333	1.0
2	0.333	0.8
3	0.333	0.6
4	0.333	0.333

The frequency response is obtained from the following equation:

$$\frac{\theta_0}{\theta_i} = \frac{4 \times 0.539(s + 0.333)}{(s + 2.38)(s + 0.333 + 0.44i)(s + 0.333 - 0.44i)} \qquad (33)$$

Figure 7 shows the individual terms of Equation (33) and the response curve with the details of construction not indicated. The term $s + 0.333$ is fixed by the system and cannot be changed unless the transmitter time constant is changed. The term $1/(s + 2.38)$ has little effect upon the response; consequently small changes in its location due to changing controller parameters will not change the response curve. The second-order curve can be changed in shape by changing the damping coefficient. As the damping coefficient decreases, the curve develops a higher peak, and when it is 0.707 there is no peak at all. As ω_n at constant ζ increases, the curve moves toward the right with no change in shape. It is possible then to modify the frequency-response curve somewhat by varying the controller parameters.

The frequency-response specifications will depend upon the process requirements. In general θ_0/θ_i should not have a high maximum because then any extraneous signals (noise) in the controller may cause a large error on the controlled variable.

The frequency response with the pressure-drop sinusoidal is obtained from the equation

$$\frac{\theta_0}{L} = \frac{s(s + 1)(s + 2)}{(s + 2.38)(s + 0.333 + 0.44i)(s + 0.333 - 0.44i)} \qquad (34)$$

The response curve is high and remains high for all frequencies, which of course is very undesirable.

A study of the figures shows that the responses are not entirely satisfactory. With proportional and integral control there is little that can be done to improve the responses. If $1/T_i$ is made much smaller, a sluggish response is obtained. As previously discussed, the damping coefficient cannot be changed sufficiently to cause much change in the responses. The addition of the integral mode of control has eliminated the steady state error but otherwise has not greatly affected or improved the response.

THREE-MODE CONTROL

The previous sections have shown that for the flow system being discussed proportional or proportional plus integral control has not proved to be satisfactory. The addition of the rate mode provides another possibility of improving the responses to both step and sinusoidal inputs.

The controller transfer function for the three-mode control is given by Equation (5). The zeros at $-z_1$ and $-z_2$ are

$$z_1, z_2 = \frac{1}{2T_r}\left(1 \pm \sqrt{1 - 4T_r/T_i}\right) \qquad (35)$$

The zeros are real if $4T_r/T_i < 1$ and complex if $4T_r/T_i > 1$ and are equal when $4T_r/T_i = 1$. The values which T_r and T_i may assume are limited only by values available on the controller being used. The effect of the choice of T_r and T_i will now be investigated.

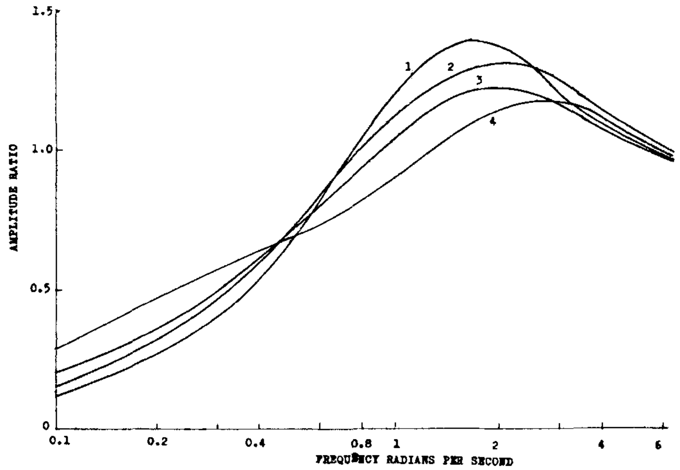

Fig. 12. Frequency response-pressure drop change.

Curve	z_1	z_2
1	0.333	1.0
2	0.333	0.8
3	0.333	0.6
4	0.333	0.333

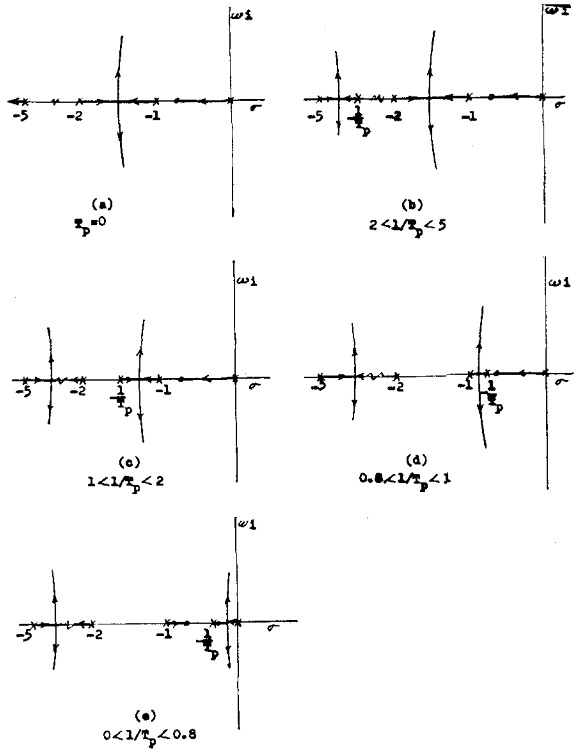

Fig. 13. Root-loci diagrams with a process lag.

TABLE 1. SUMMARY OF CALCULATIONS

z_1	0.333	0.333	0.333	0.333
z_2	1.0	0.8	0.6	0.333
T_r	0.75	0.882	1.07	1.5
T_i	4.0	4.24	4.66	6.06
For the closed-loop with $\zeta = 0.707$				
$\sigma = \omega$	0.81	0.962	1.08	1.16
p_3	—	0.583	0.38	0.19
p_4	5.38	5.49	5.46	5.49
K_c	1.41	1.26	1.08	0.84
K_cT_r	1.06	1.12	1.16	1.26
ω_n	1.145	1.356	1.53	1.643

From Equations (17) and (18) the product

$$\frac{\theta_0}{E} \times \frac{F}{\theta_0} = \frac{6.67K_cT_r(s + z_1)(s + z_2)}{s(s + 0.333)(s + 1)(s + 2)(s + 5)} \quad (36)$$

A comparison of Equation (36) with Equation (27) shows that the addition of the rate mode has added another zero and the term T_r in the numerator. The additional zero can be used to counteract the effect of another of the poles just as the single zero was used with two-mode control. The addition of T_r has the effect of changing the effective gain of the open-loop function. The total gain is K_cT_r, which may be greater of less than K_c.

The closed-loop equations now become

$$\frac{\theta_0}{\theta_i} = \frac{4K_cT_r(s + 0.333)(s + 5)(s + z_1)(s + z_2)}{s(s + 0.333)(s + 1)(s + 2)(s + 5) + 6.67K_cT_r(s + z_1)(s + z_2)} \quad (37)$$

$$\frac{\theta_0}{L} = \frac{s(s + 0.333)(s + 1)(s + 2)(s + 5)}{s(s + 0.333)(s + 1)(s + 2)(s + 5) + 6.67K_cT_r(s + z_1)(s + z_2)} \quad (38)$$

There are now two arbitrary parameters z_1 and z_2 which may be selected. The values selected determine the zeros in Equation (36), which in turn fix the location of the root-loci curves. A typical root-locus curve is shown in Figure 8 for $z_1 = 0.333$ and $z_2 = 0.8$. It was shown in the last section that as the zero moves away from the origin the system becomes less stable. This is again illustrated by the fact that as the zeros move away from the origin the branched part of the curve moves toward the origin. At the same time when one of the zeros is near the origin, a closed-loop pole is introduced near the origin. This pole produces a term in the time response that does not come to 0 very rapidly. Therefore although the response may not be cyclic, it will be slow, and excessive time will be required for the system to come to a steady state.

A study of the approximate root-loci diagrams allows one to make reasonable selections of a few sets of controller parameters for more complete calculations. A tabulation of values which have been calculated for several sets of the controller parameters is given in Table 1.

Some choices of T_r and T_i will cause z_1 and z_2 to be conjugate complex numbers thereby producing complex zeros in Equation (36). In some systems the use of complex zeros may prove beneficial. Root-loci diagrams are shown in Figure 9 for two choices of complex zeros. When the zeros are placed at

$$-1 \pm i \; (T_r = T_i = 0.5)$$

the left-branched curves no longer go to infinity but end at the zeros. For this system this change has little significance. The right-branched curves change their location so as to intersect the imaginary axis at a high value of ω with a corresponding increase in K_c. The system then will not become unstable so easily, but the closed-loop poles are too near the origin to produce a satisfactory response. The loci curves with the zeros placed at $-0.375 \pm 0.75i(T_r = 1.33,\ T_i = 1.07)$ show that the right-branched curves now end at the zeros. Since the right-branched curves do not cross the imaginary axis, the system will be stable at very high values of K_c. However the closed-loop poles near the origin are again disadvantageous.

The responses to step changes in the control point are shown in Figure 10 for a selection of controller parameters. As might be expected, the overshoot decreases as the zeros move toward the origin, and at the same time the settling time increases.

The responses to step changes in the pressure drop across the control valve are also shown in Figure 10. The maximum deviation is high for all the curves, and the time to reach steady state increases as the zeros move toward the origin. The maximum error cannot be decreased as long as the number of closed-loop poles and zeros are equal. An increase in K_c (decreasing ζ) will cause a somewhat faster recovery but will also increase the overshoot of θ_0/θ_i, which is already beyond the desirable limit.

The frequency response with the control point being varied in a sinusoidal manner is shown in Figure 11. The maximum in the curves decreases as the zeros approach the origin, with the lowest maximum being about 1.8. This is some-

trollers will not give so close control as desired or necessary. One method of improving the control that immediately comes to mind is the addition of a process time constant. This could be accomplished by the addition of a capacitance and resistance with $T_p = RC$. The open-loop equation with three-mode control is

$$\frac{\theta_0}{E} = \frac{13.33K_cT_r(s + z_1)(s + z_2)}{T_p(s + 1)(s + 2)(s + 1/T_p)} \quad (39)$$

The ratio

$$\frac{\theta_0}{E} \times \frac{E}{\theta_0} = \frac{6.67K_cT_r(s + z_1)(s + z_2)}{T_ps(s + 0.333)(s + 1)(s + 2)(s + 5)(s + 1/T_p)} \quad (40)$$

The closed-loop equations become

$$\frac{\theta_0}{\theta_i} = \frac{4K_c(T_r/T_p)(s+0.333)(s+5)(s+z_1)(s+z_2)}{s(s+0.333)(s+1)(s+2)(s+5)(s+1/T_p)+6.67K_c(T_r/T_p)(s+z_1)(s+z_2)} \quad (41)$$

$$\frac{\theta_0}{L} = \frac{(1/T_p)s(s+0.333)(s+1)(s+2)(s+5)}{s(s+0.333)(s+1)(s+2)(s+5)(s+1/T_p)+6.67K_c(T_r/T_p)(s+z_1)(s+z_2)} \quad (42)$$

what above the recommended level of 1.2 to 1.5.

A sinusoidal variation of the pressure drop across the valve produces the responses shown in Figure 12. Again the lowest maximum is obtained with the zeros nearest the origin, although there is not a significant difference between all of the curves. Curve 4 has a high value even at low frequencies, an indication that slowly changing pressure drops will cause relatively large errors in the flow rate.

PROCESS TIME CONSTANT

The previous sections have shown that the control of fluid flow when the flow system has a 0 time constant is difficult. The common industrial con-

The addition of a process time constant produces two significant changes in the equations. The term $1/T_p$ occurs in the numerator of both θ_0/θ_i and θ_0/L. Then if T_p is greater than 1, both ratios will be decreased. With large values of T_p the ratio θ_0/θ_i will be very small, which is undesirable because the system then would not respond to control-point changes. On the other hand, a large value of T_p will cause θ_0/L to be small, which is desirable. Obviously, some compromise must be made that will produce reasonably satisfactory responses of both ratios.

The second change in the equations is that an additional pole has been added to both ratios. This is significant for the ratio θ_0/L, since now the equation has an excess of one pole and the transient response curves will begin at 0 rather than 1 when $t = 0+$. Furthermore that pole at $-1/T_p$ if properly placed will produce a closed-loop pole which will partially cancel the effect of the zero at -0.333.

In a few minutes it is possible to sketch root-loci diagrams for many combinations of $1/T_p$, z_1, and z_2. Such a study will soon give one a good idea of approximate locations of these parameters for satisfactory responses. Space does not permit a complete discussion, but a brief summary follows. One zero is placed at -0.333 to cancel the pole at this point. The other zero is placed at approximately -0.8 so that the closed-loop pole will be in the neighborhood of -0.5, thereby partially canceling the effect of the zero at -0.333. Figure 13 shows the effect of the position of the pole at $-1/T_p$ upon the root-loci curves. When $1/T_p$ is larger than 2, the effect is very small as is seen by comparing Figures 13*a* and *b*. From Figures 13*b*, *c* and *d* it is seen that as $1/T_p$ becomes smaller, the branched curve approaches the origin. The branched curve should not be too near the origin; therefore there is a limit to the amount that the process time constant can be increased. It appears that letting $1/T_p$ have a value in the neighborhood of 2 will produce the best results. The responses to step inputs for several combinations of the parameters are shown in Figures 14 and 16. The differences are obvious. If one is interested, more calculations can easily be made. The corresponding frequency responses are shown in Figures 15 and 17.

All the curves show that the choice of $1/T_p = 1.8$ is approximately the best. Slight variations may improve the responses slightly. As seen from the curves, when $1/T_p$ becomes very small the responses are sluggish, requiring a long time for the system to reach the new steady state.

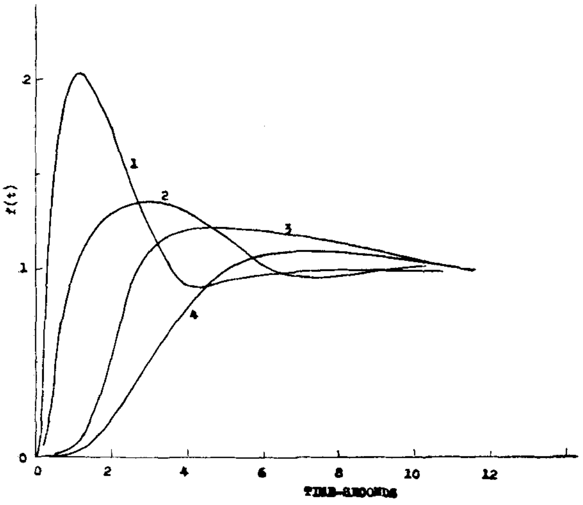

Fig. 14. Transient response after a step change in the control point.

Curve	z_1	z_2	$1/T_p$
1	0.333	0.8	∞
2	0.333	0.8	1.8
3	0.333	0.8	0.8
4	0.333	0.1	0.1

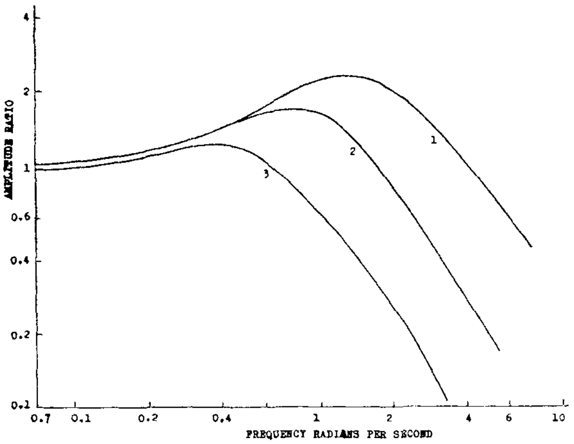

Fig. 15. Frequency response-control point change.

Curve	z_1	z_2	$1/T_p$
1	0.333	0.8	∞
2	0.333	0.8	1.8
3	0.333	0.1	0.1

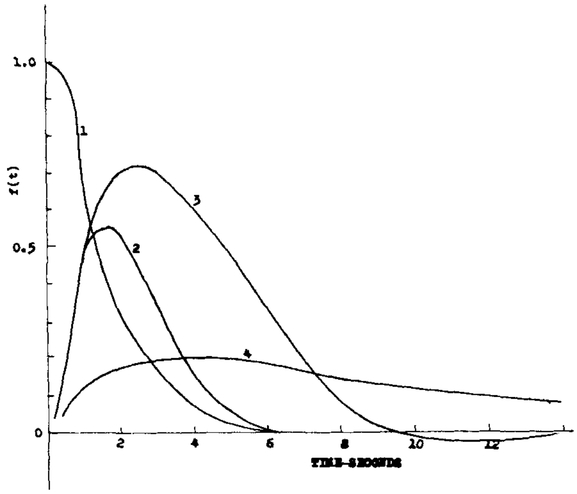

Fig. 16. Transient response after a step change in the pressure drop.

Curve	z_1	z_2	$1/T_p$
1	0.333	0.8	∞
2	0.333	0.8	1.8
3	0.333	0.8	0.8
4	0.333	0.1	0.1

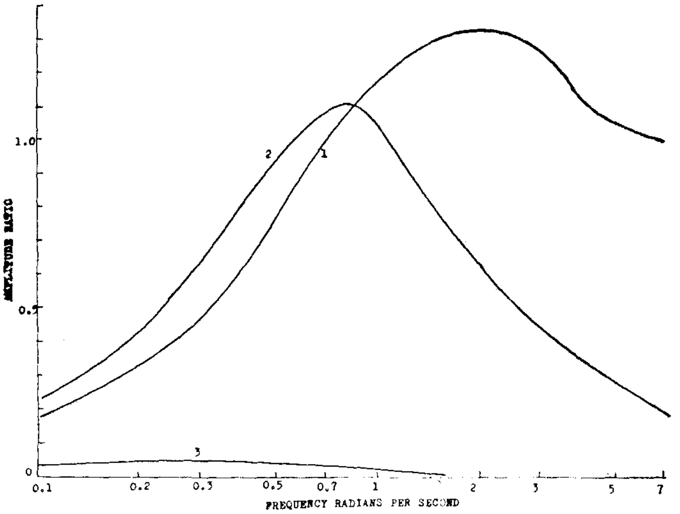

Fig. 17. Frequency response-pressure drop change.

Curve	z_1	z_2	$1/T_p$
1	0.333	0.8	∞
2	0.333	0.8	1.8
3	0.333	0.1	0.1

OTHER MODIFICATIONS OF THE SYSTEM

There are several other modifications which would improve the responses if it is found necessary to do so. As complete discussion is beyond the scope of this paper, the methods will be listed only: (a) addition of a pressure controller in the flow line (7) either as a separate controller or in cascade, (b) the design of a controller with other than the standard three modes of control, and (c) the use of an additional controller in the feedback loop or as a feed forward controller.

CONCLUSIONS

A relatively complete analysis has been made of the control of fluid flow with a flowmeter as the measuring with a flowmeter as the measuring instrument, a controller, and a control valve. The analysis is more complete than is ordinarily necessary. Anyone with a working knowledge of the methods could eliminate some of the steps or at least some of the detail. This example was carried out to illustrate the methods and to show how easily one can make an analysis of a control system. The methods of control-system analysis given have several advantages, the most important being that the engineer has a complete picture of the effect of all the system parameters upon the responses. A second important point is that much of the analysis can be made with very little calculation, particularly after one has had some experience. Obviously all this could have been done with a computer, with however the great disadvantage that one has lost completely the over-all view of the effect of the system parameters. The author believes that the procedure given is quite satisfactory for most purposes and is an excellent preliminary analysis if it is desired to investigate the nonlinear system with a computer.

Conclusions with respect to the specific control system are

1. Linearized equations for the system have been derived.
2. Proportional control is not satisfactory if the system is likely to be subjected to changing pressure drops of any great magnitude.
3. Proportional plus integral control removed the steady state error but otherwise had little effect upon the responses.
4. Three-mode control improved the responses but still not enough if very close control was wanted.
5. The addition of a process time constant of the proper value greatly improved the responses.

NOTATION

A_i = step change in reference input or control point
A_L = step change in the pressure drop across the control valve
C = capacitance
E = error signal
f = frequency cycles/sec. = (radians/sec.)/2π
F = signal
G = transfer function
h_m = pressure differential
K = constant
K_c = controller gain, (reciprocal of the proportional band/100)
L = load function, pressure drop across the valve
Δp = pressure drop across the valve
Q = flow rate
R = flow resistance
s = independent variable in transformed equations
T = time constant
z = zeros, functions of the controllers parameters

Greek Letters

ζ = damping coefficient
θ_i = reference input signal (control point)
θ_0 = controlled variable, flow rate
σ = real coordinate of the conjugate complex poles
ω = imaginary coordinate of the conjugate complex poles
ω_n = natural frequency, radians/sec.
ω_s = frequency of sinusoidal input, radians/sec.

Subscripts

c = controller
i = integral mode of control
p = process
r = rate mode of control
ss = steady state
tr = transmitter
tu = tubing
v = valve

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